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Mean Convergence of Birkhoff Interpolation Based on the Roots of Unity: A Problem of P. Turán*

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1. INTRODUCTION

The problem of the mean convergence of Birkhoff interpolation based on the roots of unity has been initiated by P. Turán [7] by posing the following problem:

PROBLEM XLVI. Is it true that, for all $f(z)$, analytic in $|z|<1$ and continuous in $|z| \leqslant 1$,

$$\lim_{n \to \infty} \int_{|z|=1} \left| f(z) - \sum_{k=1}^{n} f(e^{2\pi ik/n})\, \alpha_{m,k,0}(z) \right|^2 |dz| = 0? \tag{1.1}$$

Here $\alpha_{\bar{m},k,0}(z)$ denotes the fundamental polynomials of Birkhoff interpolation of type $\bar{m} = (0, 2)$ (where the value and second derivative of the interpolating polynomial are prescribed).

The first investigations of the Birkhoff interpolation based on the roots of unity were carried out by O. Kis [2]. This early result considered the problem of interpolating a function prescribing the value and second derivative of the interpolating polynomial at the nth roots of unity (the (0, 2) problem), and the convergence of such interpolation. After some further special cases (see Sharma [8]), Cavaretta, Sharma and Varga [1] have established the existence and uniqueness theorem for all possible interpolations of this type. A different method was provided by

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Riemenschneider and Sharma [5], and that approach was used to prove a uniform convergence theorem in the general case.

The mean convergence in the simplest case, namely the L_p-convergence of the Lagrange interpolation was proved by Lozinski [3]. Lozinski showed that if $f(z)$ analytic in $|z| < 1$ and continuous $|z| \leqslant 1$, then the Lagrange polynomial interpolant of $f(z)$ in the nth roots of unity converges to $f(z)$ in L_p-norm on the unit circle as $n \to \infty$, $p > 0$.

The object in this paper is to investigate the mean convergence of Birkhoff interpolation and extend Lozinski's result to all possible Birkhoff interpolation based on the roots of unity, i.e., to prove the generalized form of the Turán's problem.

A special case was considered by Szabados and Varma [9].

Fundamental Polynomials

Riemenschneider and Sharma [5] obtained a form of the fundamental polynomials that can be used to establish convergence theorems. Following the notations used in [5], let $\bar{m} = (m_0, m_1, \ldots, m_q)$ where $0 = m_0 < m_1 < \cdots < m_q$ are integers. Let $Z_n = \{z_k = e^{2\pi i k/n} : k = 1, \ldots, n\}$ denote the nth roots of unity.

The fundamental polynomials $\alpha_{m,k,j}(z)$ for Birkhoff interpolation based on Z_n are the unique polynomials of degree $(q+1)n - 1$ satisfying

$$\alpha_{m,k,j}^{(m_r)}(z_l) = \begin{cases} 1, & \text{if } \ r = j, \, l = k, \\ 0, & \text{otherwise}, \end{cases} \qquad 0 \leqslant r, j \leqslant q; \; 1 \leqslant k, l \leqslant n. \tag{1.2}$$

If and only if $m_j \leqslant jn$, $j = 0, \ldots, q$, the fundamental polynomials exist (see [1, 5]) and have the form

$$\alpha_{m,k,j}(z) = \sum_{\lambda=0}^{q} z^{\lambda n} P_{\lambda,k,j}(z), \tag{1.3}$$

$$P_{\lambda,k,j}(z) = \frac{(-1)^{\lambda+j}}{n} \sum_{v=0}^{n-1} \frac{M_{j,\lambda}(v)}{M(v)} z_k^{m_j - v} z^v. \tag{1.4}$$

The function $M(v)$ is defined by the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (v)_{m_1} & (v+n)_{m_1} & \cdots & (v+qn)_{m_1} \\ \vdots & \vdots & \cdots & \vdots \\ (v)_{m_q} & (v+n)_{m_q} & \cdots & (v+qn)_{m_q} \end{vmatrix},$$

where $M_{j,\lambda}(v)$ is the $(j+1, \lambda+1)$-minor of $M(v)$ $((a)_m := a(a-1) \cdots (a-m+1)$, $(a)_0 = 1)$. The determinant $M(v)$ is positive for any non-negative integer v. Let $f(z)$ be analytic for $|z| < 1$ and continuous on $|z| \leqslant 1$.

THEOREM. *Let the linear operator* $Q_{\bar{m},n}(f, z)$ *be defined by*

$$Q_{m,n}(f, z) = \sum_{j=0}^{q} \sum_{k=1}^{n} y_{k,n}^{(m_j)} \alpha_{\bar{m},k,j},$$

where

$$y_{k,n}^{m_j} := \begin{cases} f(z_{kn}), & \text{if } j=0; \\ o(n^{m_j}), & \text{uniformly in } k=1, ..., n, \text{ if } j=1, ..., q. \end{cases} \tag{1.5}$$

then

$$\lim_{n \to \infty} \|f - Q_{m,n}(f, z)\|_{L_p} = 0$$

for all positive p.

COROLLARY (See [5, Remark 2]). *The operator* $Q_{\bar{m},n}(f, z)$ *converges uniformly to* $f(z)$ *on compact subsets of* $|z| < 1$.

Remark. Turán's problem is the special case of our theorem above when $q=1$, $\bar{m}=(0, 2)$, $p=2$, and $y_{k,n}^{m_1}=0$, for $k=1, ..., n$. The theorem also includes the Hermite interpolation when $m_j=j$, $j=0, 1, ..., q$.

2. SOME LEMMAS

We first give some lemmas about special determinants arising in the proofs later on.

LEMMA 1 (See [1, 5]). *If* $0=m_0<m_1<\cdots<m_q$ *and* $m_j \leqslant jn$ *then* $M(v)>0$ *for* $v \geqslant 0$; *moreover, the generalized Vandermonde-determinant*

$$V(\alpha) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha^{m_1} & (\alpha+1)^{m_1} & \cdots & (\alpha+q)^{m_1} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha^{m_q} & (\alpha+1)^{m_q} & \cdots & (\alpha+q)^{m_q} \end{vmatrix}$$

is a positive, continuous, and increasing function of α *for* $0 \leqslant \alpha \leqslant 1$.

LEMMA 2. *The coefficients of the fundamental polynomials satisfy the following estimates*:

$$\frac{M_{j,\lambda}(v)}{M(v)} - \frac{M_{j,\lambda}(v+1)}{M(v+1)} = O\left(\frac{1}{n^{m_j+1}}\right) \qquad \text{for} \quad v=0, ..., n-1 \tag{2.1}$$

and

$$\frac{M_{j,\lambda}(n)}{M(n)} = O\left(\frac{1}{n^{m_j}}\right). \tag{2.2}$$

Proof of Lemma 2. The simple observation

$$\begin{aligned}(v+\lambda n)_{m_j} &= n^{m_j}\left(\frac{v}{n}+\lambda\right)\left(\frac{v}{n}+\lambda-\frac{1}{n}\right)\cdots\left(\frac{v}{n}+\lambda-\frac{m_j-1}{n}\right)\\ &= n^{m_j}\left(\left(\frac{v}{n}+\lambda\right)^{m_j}+O\left(\frac{1}{n}\right)\right),\end{aligned}$$

where the $O(1/n)$ term depends only on λ and $\bar{m}$, gives the estimates

$$M(v) = n^{\sum_{i=0}^{q} m_i}\left(V\left(\frac{v}{n}\right)+O\left(\frac{1}{n}\right)\right) \tag{2.3}$$

$$M_{j,\lambda}(v) = n^{\sum_{i=0, i\neq j}^{q} m_i}\left(V_{j,\lambda}\left(\frac{v}{n}\right)+O\left(\frac{1}{n}\right)\right) \tag{2.4}$$

for $0 \leqslant v \leqslant n$. Therefore, by (2.3), (2.4), and Lemma 1,

$$\frac{M_{j,\lambda}(v)}{M(v)} = \frac{1}{n^{m_j}}\,\frac{V_{j,\lambda}(v/n)+O(1/n)}{V(v/n)+O(1/n)} = O\left(\frac{1}{n^{m_j}}\right)$$

that immediately proves the second assertion of Lemma 2. Similarly, we have

$$\begin{aligned}&\frac{M_{j,\lambda}(v)}{M(v)} - \frac{M_{j,\lambda}(v+1)}{M(v+1)}\\ &\quad= \frac{1}{n^{m_j}}\left(\frac{V_{j,\lambda}(v/n)+O(1/n)}{V(v/n)+O(1/n)} - \frac{V_{j,\lambda}((v+1)/n)+O(1/n)}{V((v+1)/n)+O(1/n)}\right).\end{aligned}$$

We need only estimate the terms in parentheses. The elementary estimate

$$\left(\frac{v+1}{n}+\lambda\right)^{m_j} = \left(\frac{v}{n}+\lambda\right)^{m_j} + O\left(\frac{1}{n}\right)$$

shows that

$$V_{j,\lambda}\left(\frac{v+1}{n}\right) = V_{j,\lambda}\left(\frac{v}{n}\right) + O\left(\frac{1}{n}\right)$$

$$V\left(\frac{v+1}{n}\right) = V\left(\frac{v}{n}\right) + O\left(\frac{1}{n}\right)$$

and these, together with the estimates above, prove (2.1).

LEMMA 3 (M. Riesz [6]). *Let* $f(z)=\sum_{k=0}^{\infty} a_k z^k$ *be analytic in* $|z|<1$ *and continuous in* $|z|\leqslant 1$ *and let* $s_n(z)=\sum_{k=0}^{n} a_k z^k$ *be the* nth *partial sum of the power series expansion for* $f(z)$. *Then*

$$\|s_n(z)\|_{L_p}\leqslant c_p\,\|f(z)\|_{L_p}$$

for all $p>1$.

LEMMA 4 (Marcinkiewicz and Zygmund [4]). *For any algebraic polynomial* $P_n(z)$ *of degree at most* $n-1$, *there is a constant* c_p *depending only on* p, *such that*

$$\|P_n(z)\|_{L_p}\leqslant c_p\left(\frac{1}{n}\sum_{k=1}^{n}|P_n(z_k)|^p\right)^{1/p}\qquad \textit{for all}\quad p>1.$$

3. PROOF OF THEOREM

If $f(z)$ is analytic for $|z|<1$ and continuous for $|z|=1$, then let $\omega_f(\delta)$ be the modulus of continuity of $f(e^{ix})$. Using the lemma of O. Kis [2] there exists a polynomial $G(z)$ of degree $n(q+1)-1$ that

$$|f(z)-G(z)|\leqslant c_1\,\omega(1/n) \tag{3.1}$$

and

$$|G^{(m)}(z)|\leqslant c_2 n^m\omega(1/n)\qquad \text{for}\quad m=1,2,\ldots \tag{3.2}$$

on the closed unit circle. The unicity of the Birkhoff interpolation implies that $G(z)$ can be written in the form

$$G(z)=\sum_{j=0}^{q}\sum_{k=1}^{n} G^{(m_j)}(z_{kn})\,\bar{\alpha}_{m,k,j}.$$

Since

$$\left\|f-\sum_{j=0}^{q}\sum_{k=1}^{n} y_{k,n}^{(m_j)}\alpha_{m,k,j}\right\|_{L_p}\leqslant \|f-G\|_{L_p}+\left\|G-\sum_{j=0}^{q}\sum_{k=1}^{n} y_{k,n}^{(m_j)}\alpha_{\bar{m},k,j}\right\|_{L_p}$$

and by (3.1) $\|f-G\|_{L_p}\leqslant c_1\,\omega(1/n)$, it is enough to prove that

$$\begin{aligned}\lim_{n\to\infty}&\left\|G-\sum_{j=0}^{q}\sum_{k=1}^{n} y_{k,n}^{(m_j)}\alpha_{m,k,j}\right\|_{L_p}\\ &=\lim_{n\to\infty}\left\|\sum_{j=0}^{q}\sum_{k=1}^{n}\left(G^{(m_j)}(z_{kn})-y_{k,n}^{(m_j)}\right)\alpha_{\bar{m},k,j}\right\|_{L_p}=0.\end{aligned} \tag{3.3}$$

Substituting the explicit form of the fundamental polynomials into (3.3) and with the notation $g_{k,j,n} := G^{(m_j)}(z_{kn}) - y_{k,n}^{(m_j)}$, we need only estimate

$$\sum_{j=0}^{q} \sum_{\lambda=0}^{q} \left\| \frac{1}{n} \sum_{\nu=0}^{n-1} \frac{M_{j,\lambda}(\nu)}{M(\nu)} \sum_{k=1}^{n-1} g_{k,j,n} z_k^{m_j-\nu} z^{\nu} \right\|_{L_p}. \tag{3.4}$$

Using the summation by parts formula with respect to

$$G_j(\nu) := \sum_{r=0}^{\nu} \left(\sum_{k=1}^{n} g_{k,j,n} z_k^{m_j-r} \right) z^r, \qquad G(-1) = 0, \tag{3.5}$$

we have

$$\sum_{\nu=0}^{n-1} \frac{M_{j,\lambda}(\nu)}{M(\nu)} \sum_{k=1}^{n-1} g_{k,j,n} z_k^{m_j-\nu} z^{\nu}$$
$$= \sum_{\nu=0}^{n-1} \left(\frac{M_{j,\lambda}(\nu)}{M(\nu)} - \frac{M_{j,\lambda}(\nu+1)}{M(\nu+1)} \right) G_j(\nu) + \frac{M_{j,\lambda}(n)}{M(n)} G_j(n-1).$$

Consequently, by (3.4),

$$\left\| G - \sum_{j=0}^{q} \sum_{k=1}^{n} y_{k,n}^{(m_j)} \alpha_{m,k,j} \right\|_{L_p}$$
$$\leqslant \sum_{j=0}^{q} \sum_{\lambda=0}^{q} \left\| \frac{1}{n} \sum_{\nu=0}^{n-1} \left(\frac{M_{j,\lambda}(\nu)}{M(\nu)} - \frac{M_{j,\lambda}(\nu+1)}{M(\nu+1)} \right) G_j(\nu) \right\|_{L_p}$$
$$+ \sum_{j=0}^{q} \sum_{\lambda=0}^{q} \left\| \frac{1}{n} \frac{M_{j,\lambda}(n)}{M(n)} G_j(n-1) \right\|_{L_p} \tag{3.6}$$

Applying Lemma 2, the first term on the right in inequality (3.6) is bounded by

$$O\left(\frac{1}{n^{m_j+1}} \sum_{j=0}^{q} \max_{\nu=0,1,\dots,n-1} \|G_j(\nu)\|_{L_p} \right) \tag{3.7}$$

Similarly for the second term in (3.6), we obtain

$$O\left(\frac{1}{n^{m_j}} \sum_{j=0}^{q} \left\| \frac{1}{n} G_j(n-1) \right\|_{L_p} \right). \tag{3.8}$$

Since the polynomials

$$G_j(\nu) = \sum_{r=0}^{\nu} \left(\sum_{k=1}^{n} g_{k,j,n} z_k^{m_j-r} \right) z^r, \qquad \nu = 0, 1, \dots, n-1$$

are the partial sums of the power series expansion for $G_j(n-1)$, it follows from Lemma 3 that

$$\|G_j(\nu)\|_{L_p} \leqslant_p \|G_j(n-1)\|_{L_p} \qquad \text{for} \quad \nu = 0, 1, ..., n-1. \tag{3.9}$$

Combining the inequalities of (3.6)–(3.9), we obtain

$$\left\| G - \sum_{j=0}^{q} \sum_{k=1}^{n} y_{k,n}^{(m_j)} \alpha_{m,k,j} \right\|_{L_p} \leqslant O\left(\frac{1}{n^{m_j}} \sum_{j=0}^{q} \left\| \frac{1}{n} G_j(n-1) \right\|_{L_p} \right). \tag{3.10}$$

In order to apply Lemma 4 for the polynomials $G_j(n-1)$, we need to compute their values on the nth roots of unity. Recalling the definitions of (3.5), we see that

$$\frac{1}{n} G_j(n-1) = \frac{1}{n} \sum_{r=0}^{n-1} \left(\sum_{k=1}^{n} g_{k,j,n} z_k^{m_j - r} \right) z^r = \sum_{k=1}^{n} g_{k,j,n} z_k^{m_j} \frac{1}{n} \sum_{r=0}^{n-1} z_k^{-r} z^r$$

and

$$\frac{1}{n} G_j(n-1)|_{z=z_k} = g_{k,j,n} z_k^{m_j}.$$

Thus, by Lemma 4,

$$\left\| \frac{1}{n} G_j(n-1) \right\|_{L_p} \leqslant c_p \left(\frac{1}{n} \sum_{k=1}^{n} |g_{k,j,n} z_k^{m_j}|^p \right)^{1/p}.$$

We remark from (3.2) and (1.5) that $g_{k,j,n} = o(n^{m_j})$ uniformly in $k = 1, ..., n$; $j = 0, ..., q$; therefore,

$$\left\| \frac{1}{n} G_j(n-1) \right\|_{L_p} = o(n^{m_j}).$$

Then, substituting the above in (3.10) gives

$$\left\| G - \sum_{j=0}^{q} \sum_{k=1}^{n} y_{k,n}^{(m_j)} \alpha_{m,k,j} \right\|_{L_p} = o(1)$$

that completes the proof of the theorem. Finally, by the Cauchy integral formula, the corollary is an immediate consequence of the theorem.

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References

1. A. S. Cavaretta, A. Sharma, and R. S. Varga, Hermite–Birkhoff interpolation in the nth roots of unity, *Trans. Amer. Math. Soc.* **259** (1980), 621–628.
2. O. Kis, Notes on interpolation, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 49–64.
3. S. M. Lozinski, Über interpolation, *Mat. Sb. (N.S.)* **8**, No. 50 (1940), 57–68.
4. J. Marcinkiewicz and A. Zygmund, Mean values of trigonometrical polynomials, *Fund. Math.* **28** (1937), 131–166.
5. S. D. Riemenschneider and A. Sharma, Birkhoff interpolation at the nth roots of unity: convergence, *Canad. J. Math.* **33** (1981), 362–371.
6. M. Riesz, Sur les fonctions conjuguées, *Math. Z.* **27** (1928), 218–244.
7. P. Turán, On some open problems of approximation theory, *J. Approx. Theory* **29** (1980), 23–85.
8. A. Sharma, Some poised and nonpoised problems on interpolation, *SIAM Rev.* **14** (1972), 129–151.
9. J. Szabados and A. K. Varma, On an open problem of P. Turán concerning Birkhoff interpolation based on the roots of unity, *J. Approx. Theory* **47** (1986), 255–264.